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ON A PROBLEM OF A. Yu. ISHLINSKII†

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The restricted four-body problem (point masses) is considered, in order to investigate whether two small bodies can move in quasi-circular orbits in the gravitational field of two large bodies. This problem may be of interest in the context of the dynamics of asteroid systems. © 2003 Elsevier Ltd. All rights reserved.

1. A. Yu. ISHLINSKII'S FORMULATION OF THE PROBLEM

In a talk with the present author, A. Yu. Ishlinskii posed the following problem (Fig. 1): "Two bodies, each of the same mass M, attracting one another, are moving in circular orbits. Suddenly, two small bodies appear, each of the same mass m, which have initial conditions such that, were it not for the bodies M, they would also move in a circular orbit. The interaction is Newtonian. What disturbances will the bodies M introduce into the motion of the bodies m?"

2. FORMULATION OF THE PROBLEM

Let us assume that $m \ll M$ and formulate the restricted problem of celestial mechanics, assuming that the bodies m attract one another and the bodies M but do not affect the circular motion of the bodies M; the bodies M also attract one another.

Let the distance between the bodies M be a. Expressing all distances in units of a, we introduce the dimensionless time $\tau = \omega t$, where $\omega = \sqrt{2Mf}a^{-3/2}$ is the angular velocity of the bodies M. In the dimensionless variables thus introduced, the equations of motion in a system of coordinates rotating together with the bodies (see Fig. 1) have the form (the two-dimensional case)

$$x_i'' - 2y_i' = \frac{\partial W}{\partial x_i}, \quad y_i'' + 2x_i' = \frac{\partial W}{\partial y_i}, \quad i = 1, 2 \tag{2.1}$$

$$W = \frac{1}{2}\sum_{i=1}^{2}(x_i^2 + y_i^2) + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{1}{r_{i1}} + \frac{1}{r_{i2}}\right) + \frac{\alpha}{2}\frac{1}{\rho}, \quad \alpha = \frac{m}{M} \tag{2.2}$$

where r_{i1} and r_{i2} are the distances from the small bodies ($i = 1, 2$) to the large ones, and ρ is the distance between the small bodies. The system of equations (2.1) is of the eighth order – a problem with four degrees of freedom.

3. SYMMETRIC TRAJECTORIES

However, we shall not consider the general problem (2.1), (2.2), but the problem formulated in Section 1. Somewhat generalizing that formulation, we shall consider trajectories which satisfy symmetric initial conditions

$$\mathbf{r}_1^0 = -\mathbf{r}_2^0, \quad \dot{\mathbf{r}}_1^0 = -\dot{\mathbf{r}}_2^0 \tag{3.1}$$

where $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$ are the initial positions of the small bodies relative to the origin, and $\dot{\mathbf{r}}_1^0$ and $\dot{\mathbf{r}}_2^0$ are their respective velocities.

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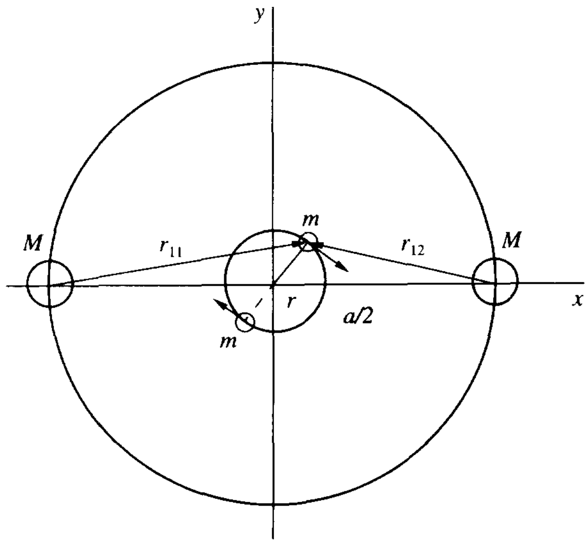

Fig. 1

It follows directly from the equations of motion (2.1), (2.2) that symmetric trajectories exist satisfying conditions (3.1) and Eqs (2.1) and (2.2)

$$\mathbf{r}_1(t) = -\mathbf{r}_2(t) \tag{3.2}$$

In what follows we shall concern ourselves only with such trajectories.

For symmetric trajectories (3.2) the order of system (2.1) is halved (the problem with two degrees of freedom). The equations of motion become

$$x'' - 2y' = \partial W/\partial x, \quad y'' + 2x' = \partial W/\partial y$$

$$W = \frac{r^2}{2} + \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{\beta}{r}, \quad \beta = \frac{\alpha}{8} \tag{3.3}$$

$$r = \sqrt{x^2 + y^2}, \quad r_1 = \sqrt{(x - 1/2)^2 + y^2}, \quad r_2 = \sqrt{(x + 1/2)^2 + y^2}$$

where r is the dimensionless distance from the origin to one of the "small" point masses, r_1 and r_2 are their respective distances to the "large" point masses.

Equations (3.3) have a first integral

$$\frac{1}{2}(x'^2 + y'^2) - W = -c$$

from which it follows that the actual motion takes place in the domain

$$W \geq c$$

and is bounded by surfaces of zero velocity $W = c$.

Before constructing the surfaces of zero velocity, we will consider the points of libration of problem (3.3). These points correspond to solutions of the system of equations

$$\partial W/\partial x = 0, \quad \partial W/\partial y = 0$$

or, in explicit form:

(1) the points L_1 and L_2 with coordinates $y_1 = y_3 = 0, x_1 = -x_3$, where x_3 $(x_3 > 1/2)$ satisfy the equation

$$x_3 - (x_3^2 + 1/4)/(x_3^2 - 1/4)^2 - \beta/x_3^2 = 0$$

(2) the points L_2^+ and L_2^- with coordinates $y_2^+ = y_2^- = 0, x_2^- = -x_2^+ = -x^*$, where x_* $(0 < x_* < 1/2)$ satisfy the equation

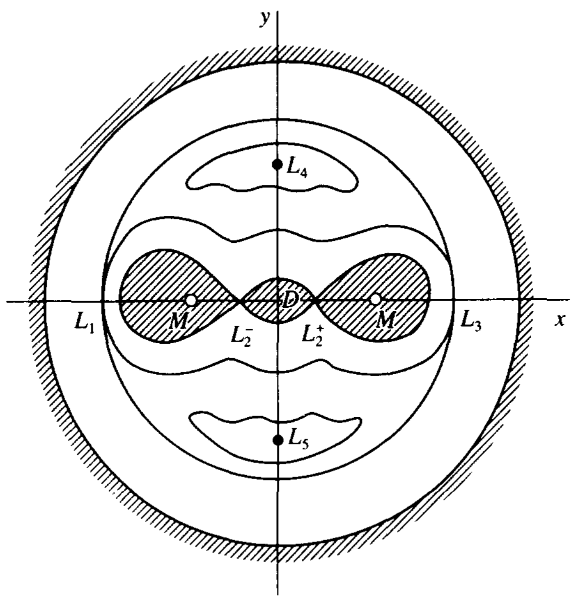

Fig. 2

$$x_*^3(1 + (1/4 - x_*^2)^2) = \beta(1/4 - x_*^2)^2$$

(3) the points L_4 and L_5 with coordinates $x_4 = x_5 = 0, y_5 = -y_4$, where y_4 ($y_4 > 0$) satisfy the equation

$$y_4^2((1/4 + y_4^2)^{3/2} - 1) = \beta(1/4 + y_4^2)^{3/2}$$

whence, among other things, it follows that $y_4 > \sqrt{3}/2$.

Where $\beta = 0$ Eqs (3.3) become the equations of the restricted three-body problem where the two attracting bodies have the same mass. Correspondingly, the points L_2^+ and L_2^- contract as $\beta \to 0$ to the point $L_2(0, 0)$ and the points L_i^- ($i = 1, \ldots, 5$) become the points of libration of the restricted three-body problem (for example, $y_4 > \sqrt{3}/2$, etc.).

The zero velocity surface $W = c$ of problem (3.3) is shown qualitatively in Fig. 2. As an example, the domain of possible motions $W \geqslant c_2$ is shown hatched, where c_2 is a value of the constant c such that the surface $W = c_2$ passes through the points of libration L_2^+ and L_2^-.

If $c > c_2$, the motion never leaves some neighbourhood D of the origin, or does not leave a certain neighbourhood of one of the points M, or, at very large initial distances from the origin, always remains at a considerable distance from it.

If c is somewhat smaller than c_2, then a point, having begun to move in the neighbourhood of the origin, may proceed, through a "throat" in the neighbourhood of L_2^+ (L_2^-), to move in the neighbourhood of one of the points M, then the reverse, etc.

A value of $c = c_{13}$ exists such that the surface $W = c_{13}$ passes through the points of libration $L_{1,3}$. If $c < c_{13}$, the domain of possible motions $W > c$ is unbounded: motion through the "throat" in neighbourhood of the points $L_{1,3}$ may depart as far from the origin as desired, even if it began in the neighbourhood of the origin. If $c = c_{13}$, the parts of the surface $W = c$ merge at the points L_1 and L_3.

The surface $W = c_2$ is of crucial importance for our problem. The initial motion remains in the neighbourhood D of the origin if it began there with initial conditions such that $c \geqslant c_2$. The motion will then never leave the domain D: $W \geqslant c_2$ (Fig. 2). This domain is bounded by a certain oval on which the largest value of the coordinates x is $x_* = L_2^+$. Approximate calculation shows that

$$x_* \approx (\beta/17)^{1/3} \approx 0.389\beta^{1/3} \tag{3.4}$$

In that case

$$c_2 = \frac{x_*^2}{2} + \frac{2}{1 - 4x_*^2} + \frac{\beta}{x_*} \tag{3.5}$$

or, approximately

$$c_2 \approx 2 + \frac{3}{2}(17\beta^2)^{1/3} \tag{3.6}$$

and the maximum of the coordinate y_* on the boundary of the domain is determined by the conditions

$$y_* \approx x\beta^{1/3}; \quad \frac{1}{x} - \frac{7}{2}x^2 = \frac{3}{2}17^{1/3}$$

so that

$$y_* = 0.245\beta^{1/3} \tag{3.7}$$

Thus, we have the following partial answer to the question posed in Section 1.

If the symmetric initial conditions are such that the value of the constant of the Jacobi integral is $c \geqslant c_2$, where c_2 is defined by (3.5) or (3.6), then the motion will always remain in the neighbourhood D of the origin defined by D: $W \geqslant c \geqslant c_2$. The maximum dimensions of the domain D are determined by the dimensional values of the coordinates x, y:

$$x_{\max} \approx 0.1945a\alpha^{1/3}, \quad y_{\max} \approx 0.1225a\alpha^{1/3}; \quad \alpha = m/M \tag{3.8}$$

As an approximation, we can assume that the domain D is an ellipse with semi-axes $x_{\max}$ and $y_{\max}$. This is the situation for any symmetric (not necessarily circular) initial data.

4. THE CASE OF CIRCULAR INITIAL DATA

It will now be useful to consider the situation for circular initial data, corresponding to the original formulation of the problem (Section 1). In that case

$$v_0^2 = \left(\sqrt{\frac{\beta}{r_0}} \mp r_0\right)^2 \tag{4.1}$$

and accordingly

$$c_\pm = \frac{1}{2}\left(\frac{1}{r_{10}} + \frac{1}{r_{20}}\right) + \frac{1}{2}\frac{\beta}{r_0} \pm \sqrt{\beta r_0} \tag{4.2}$$

The upper sign corresponds to "forward" motion and the lower one to "reverse" motion (relative to the direction in which the points M are moving). The necessary conditions for the motion to be bounded (motion in the domain D) are

$$c_\pm \geq c_2 \tag{4.3}$$

All other conditions being equal, the "reverse" motion is "less stable" than forward motion, since in reverse motion condition (4.3) is readily violated because $c_- < c_+$.

Up to the leading terms of the expansion

$$c_\pm \approx 2 + 8x_0^2 - 4y_0^2 + \frac{1}{2}\frac{\beta}{r_0} \pm \sqrt{\beta r_0} \tag{4.4}$$

Taking relations (3.6), (4.3) and (4.4) into consideration, we observe that the domain in which condition (4.3) is satisfied is bounded by the surface (or surfaces)

$$8x_0^2 - 4y_0^2 + \frac{1}{2}\frac{\beta}{r_0} \pm \sqrt{\beta r_0} = \frac{3}{2}(17\beta^2)^{1/3} \tag{4.5}$$

Defining

$$x_0 = \bar{x}_0\beta^{1/3}, \quad y_0 = \bar{y}_0\beta^{1/3}$$

from Eqs (4.5) we obtain equations in the normalized variables $\bar{x}_0$ and $\bar{y}_0$:

$$8\bar{x}_0^2 - 4\bar{x}_0^2 + \frac{1}{2\bar{r}_0} \pm \sqrt{\bar{r}_0} = \frac{3}{2}17^{1/3} \tag{4.6}$$

Estimative investigation of the surfaces (4.5) shows that they are ovals with the following semi major axes:

for forward motion

$$x_{0\max} \approx 0.38x_*, \quad y_{0\max} \approx 0.36x_* \tag{4.7}$$

for reverse motion

$$x_{0\max} \approx 0.31x_*, \quad y_{0\max} \approx 0.29x_* \tag{4.8}$$

The value of x_* is defined by (3.4).

Thus, if the initial data are circular and the motion is linear, it will never leave the domain D (an oval with semi-axes (3.8)), provided that the initial coordinates lie within the surface (4.5) – with the plus sign before the radical on the left. This surface is an oval with semi-axes

$$x_{\max}^{+} = 0.074a\alpha^{1/3}, \quad y_{\max}^{+} = 0.070a\alpha^{1/3} \tag{4.9}$$

An analogous conclusion holds for reverse circular motions, except that the semi-axes of the corresponding oval have values

$$x_{\max}^{-} = 0.060a\alpha^{1/3}, \quad y_{\max}^{-} = 0.056a\alpha^{1/3} \tag{4.10}$$

5. WEAKLY PERTURBED MOTION

Let $a/2 = R$ be the distance from the origin to one of the points M (see Fig. 1). The domain D is an oval with semi-axes

$$x_{\max} = 0.389R\alpha^{1/3}, \quad y_{\max} = 0.245R\alpha^{1/3} \tag{5.1}$$

We introduce two spheres with radii

$$r_+ = 0.140R\alpha^{1/3}, \quad r_- = 0.112R\alpha^{1/3} \tag{5.2}$$

It is clear from the foregoing that, if that initial motion is circular forward motion and the initial value is $r = r_0 \leqslant r_+$, then the motion will always take place inside the domain D. We shall call such motion weakly perturbed. But if $r_0 > r_+$ (more precisely, if $r_0 > 0.148R\alpha^{1/3}$), then "anything can happen": the motion may leave the domain D, move into a neighbourhood of the points M, turn back, etc.

If the motion is not weakly perturbed, we shall say that it is strongly perturbed.

In exactly the same way, reverse circular motion will be weakly perturbed for $r_0 \leqslant r_-$ and strongly perturbed for $r_0 > 0.120R\alpha^{1/3}$ (it is implicitly assumed that r_0 is not too large: it is quite obvious that, if $r_0 \gg R$, circular motion will be weakly perturbed, as in the "external version" of the restricted three-body problem).

Weakly perturbed motion is conveniently studied using, instead of the function W, a certain equivalent function containing only the leading terms of the expansions of the quantities r_1^{-1} and r_2^{-1}. Apart from an additive constant, one then has

$$W = \frac{17}{2}x^2 - \frac{7}{2}y^2 + \frac{\beta}{r} \tag{5.3}$$

The equations of motion become

$$x'' - 2y' = 17x - \frac{\beta x}{(x^2+y^2)^{3/2}}, \quad y'' + 2x' = -7y - \frac{\beta y}{(x^2+y^2)^{3/2}} \tag{5.4}$$

Normalizing the Cartesian coordinates

$$x = \bar{x}\beta^{1/3}, \quad y = \bar{y}\beta^{1/3}$$

we reduce Eqs (5.4) to the form

$$x'' - 2y' + \left(\frac{1}{r^3} - 17\right)x = 0, \quad y'' + 2x' + \left(\frac{1}{r^3} + 7\right)y = 0; \quad r^3 = (x^2+y^2)^{3/2} \tag{5.5}$$

where, for brevity, we have omitted the bar in the notation of the normalized coordinates. Note that in weakly perturbed motion it is invariably true that $1/r^3 > 17$.

Changing in Eqs (5.5) to polar coordinates

$$x = r\cos\phi, \quad y = r\sin\phi$$

we obtain

$$r'' - r(\phi' + 1)^2 + \frac{1}{r^2} = r(16\cos^2\phi - 8\sin^2\phi)$$
$$\frac{d}{dt}r^2(\phi' + 1) = -24r^2\cos\phi\sin\phi \tag{5.6}$$

Of course, Eqs (5.6) have a first integral

$$\frac{1}{2}(r'^2 + r^2\phi'^2) - \frac{1}{r} - \frac{17}{2}r^2\cos^2\phi + \frac{7}{2}r^2\sin^2\phi = -c \tag{5.7}$$

The quantity $\phi' + 1$ is the angular velocity in absolute motion. Since we are considering weakly perturbed motion, it follows that ϕ is a fast phase ($|\phi'| \gg 1$) and, to a certain approximation, system (5.6) can be replaced by the system averaged over the fast phase

$$r'' - r(\phi' + 1)^2 + \frac{1}{r^2} = 4r, \quad r^2(\phi' + 1) = r_0^2\omega_0 \tag{5.8}$$

where r_0 is the radius of the unperturbed orbit and ω_0 is the absolute angular velocity in the unperturbed orbit (which may have a plus or minus sign).

It follows from system (5.8) (noting that $\omega_0^2 = 1/r_0^3$) that

$$r'' + \frac{1}{r^2} - \frac{r_0}{r^3} - 4r = 0 \tag{5.9}$$

By (5.9), a small deviation δr from the initial circular orbit will satisfy the equation

$$(\delta r)'' + \Omega^2\delta r = 4r_0, \quad \Omega^2 = \omega_0^2 - 4 = \frac{1}{r_0^3} - 4 \tag{5.10}$$

Perturbations and motions due only to the effect of the outer masses M are evaluated by integrating Eqs (5.10) with initial conditions $\delta r_0^o = 0$, $\delta r_0' = 0$. It then turns out that

$$\delta r = \frac{8r_0}{\Omega^2}62\sin\frac{\Omega}{2}r \tag{5.11}$$

Of course, this formulation only holds over a bounded time interval.

Formula (5.11) describes very small perturbations: $\delta r/r_0 \approx 10^{-3}$. At the same time, it follows from accurate estimates that the condition $\delta r/r_0 \approx 1$ is not violated.

Remarks. 1. The above analysis only describes the situation within the set of symmetric trajectories, without touching no more general questions, such as the stability or instability, in some sense, of the set of symmetric trajectories to arbitrary asymmetric perturbations of the initial data.

2. This paper was first published in a collection of papers [1] which is accessible only with difficulty. In the light of recent astronomical discoveries of systems of dual asteroids [2] (not excluding systems with a larger number of asteroids), the papers has acquired renewed interest.

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REFERENCES

1. BELETSKII, V. V., On a restricted three-body problem. In *Almost Periodic Orbits in Celestial Mechanics*, ed. Ye. P. Aksenov. Izd. Mosk. Gos. Univ., Moscow, 1990, pp. 83–92.
2. FINKEL'SHTEIN, A. M., *The Russian Academy of Sciences between Mars and Jupiter*. Nauka, St Petersburg, 2001.

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